Composites in the Ulam Spiral

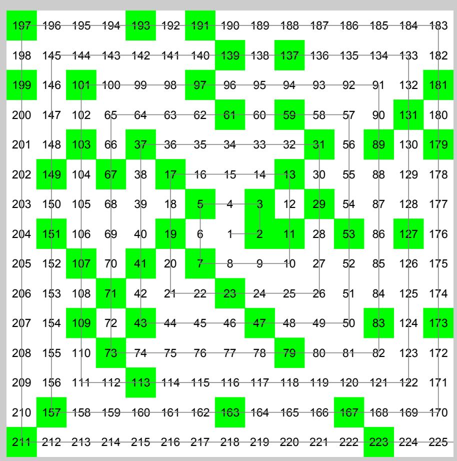
In 1963, Stanisław Ulam plotted the natural numbers in a rectilinear spiral array and observed that prime values in the resulting *Ulam spiral* tend to cluster on certain diagonal lines (this phenomenon can be explained with the Bateman–Horn conjecture). On the other hand, we claim that the Ulam spiral contains arbitrarily large square patches consisting entirely of composite numbers.

Suppose that $f_1, f_2, \ldots, f_k \in \mathbb{Z}[x]$ are nonconstant with positive leading coefficients. If $f_i(0) = 0$ for some i, then $f_i(n)$ is composite if $n \in \mathbb{N}$ is composite. Without loss of generality, suppose that $f_1(0), f_2(0), \ldots, f_k(0) \neq 0$ and let $\ell = \operatorname{lcm}\{f_1(0), f_2(0), \ldots, f_k(0)\}$. Then $f_i(j\ell) \equiv f_i(0) \equiv 0 \pmod{f_i(0)}$ for $i = 1, 2, \ldots, k$ and $j \in \mathbb{N}$. If $j \in \mathbb{N}$ is sufficiently large, then $f_i(j\ell) > f_i(0)$ and hence $f_i(j\ell)$ is composite (it has $f_i(0)$ as a proper divisor).

Observe that the nth element on the diagonal ray $1, 9, 25, 49, \ldots$ in the Ulam spiral is $(2n-1)^2$. The $d \times d$ block in the Ulam spiral with $(2n-1)^2$ as its upper-left corner is the matrix $A(n)$ given by

$$\begin{bmatrix} (2n-1)^2 & (2n+1)^2+1 & \cdots & (2(n+d)-3)^2+(d-1) \\ (2n+1)^2-1 & (2n+1)^2 & \cdots & (2(n+d)-3)^2+(d-2) \\ \vdots & \vdots & \ddots & \\ (2(n+d)-3)^2-(d-1) & (2(n+d)-3)^2-(d-2) & \cdots & (2(n+d)-3)^2 \end{bmatrix}.$$

Since this is an array of nonconstant polynomials with positive leading coefficients, there exists an $m \in \mathbb{N}$ such that each entry of $A(m)$ is composite. Thus, the Ulam spiral contains arbitrarily large square patches of composite numbers.



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